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ON THE BICYCLE PROBLEM

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We present a counterexample to a conjecture of Shinji Masuda.

The *bicycle problem*, studied by S. Masuda [1], involves a team of n people with a single one-seat bicycle travelling from a fixed origin to a fixed destination at a distance d: given the walking speed w_i and the bicycling speed b_i of each person i, one aims to minimize the arrival time of the last person at the destination. An interesting feature of this problem is the fact that an optimal schedule may require bicycling in the reverse direction. For instance, if $n=3$, $d=100$ and

$$w_1=1,\quad b_1=6,\quad w_2=2,\quad b_2=8,\quad w_3=1,\quad b_2=6,$$

then an optimal schedule goes as follows: the first person

bicycles from 0 to 54 in 9 units of time and
walks from 54 to 100 in 46 units of time,

the second person

walks from 0 to 54 in 27 units of time,
bicycles from 54 to 46 in 1 unit of time and
walks from 46 to 100 in 27 units of time,

the third person

walks from 0 to 46 in 46 units of time and
bicycles from 46 to 100 in 9 units of time.

Masuda observed that the arrival time of the last person is always bounded from below by the optimal value t^* of the linear programming problem

$$\begin{aligned} &\text{minimize } t \text{ subject to} \\ &x_i+u_i+y_i+z_i\le t \qquad (i=1,2,\dots,n), \\ &w_ix_i-w_iu_i+b_iy_i-b_iz_i=d \qquad (i=1,2,\dots,n), \\ &\sum_{i=1}^n y_i+\sum_{i=1}^n z_i\le t, \qquad (1) \end{aligned}$$

$$\sum_{i=1}^{n} b_i y_i - \sum_{i=1}^{n} b_i z_i \le d,$$
$$x_i, u_i, y_i, z_i \ge 0 \qquad (i = 1, 2, \ldots, n).$$

To justify this claim, it suffices to interpret

x_i as the time person i spends walking forward,
u_i as the time person i spends walking backward,
y_i as the time person i spends bicycling forward,
z_i as the time person i spends bicycling backward.

(In particular, the inequality $\sum b_i y_i - \sum b_i z_i \le d$ follows from the fact that the team may be confined to the interval $[0, d]$: all movements outside this interval may be replaced by resting at its boundary.)

In fact, Masuda conjectured that (under the natural assumption that $b_i > w_i$ for all i) there is always a schedule with the arrival time of the last person equal to t^*. We shall settle this conjecture by

(i) constructing, for every positive ε, a schedule in which everybody arrives at the destination before time $t^* + \varepsilon$ (this can be done even without the assumption that $b_i > w_i$ for all i), and

(ii) exhibiting data (such that $b_i > w_j$ for all i and j) for which there is no schedule with the last person arriving at the destination at time t^*.

(After this paper was written, David Gale informed me that Raphael Robinson established (i) earlier by an argument similar to the one presented here.)

Masuda's analysis of (1) begins by the observation that every optimal solution x_i, u_i, y_i, z_i, t^* induces an optimal solution $x_i^*, u_i^*, y_i^*, z_i^*, t^*$ such that $u_i^* = 0$ and $y_i^* z_i^* = 0$ for all i. (The proof is simple: it suffices to set

$$x_i^* = x_i - u_i, \qquad u_i^* = 0, \qquad y_i^* = \max(y_i - z_i, 0), \qquad z_i^* = \max(z_i - y_i, 0)$$

whenever $x_i \ge u_i$ and

$$x_i^* = u_i^* = 0, \qquad y_i^* = d/b_i, \qquad z_i^* = 0$$

whenever $x_i < u_i$.) To put it differently, t^* is the optimal value of the problem

$$\begin{aligned}
&\text{minimize } t \text{ subject to}\\
&x_i + y_i + z_i \le t \qquad (i = 1, 2, \ldots, n),\\
&w_i x_i + b_i y_i - b_i z_i = d \qquad (i = 1, 2, \ldots, n),\\
&\sum_{i=1}^{n} y_i + \sum_{i=1}^{n} z_i \le t,\\
&\sum_{i=1}^{n} b_i y_i - \sum_{i=1}^{n} b_i z_i \le d,\\
&x_i, y_i, z_i \ge 0 \qquad (i = 1, 2, \ldots, n),\\
&y_i z_i = 0 \qquad (i = 1, 2, \ldots, n).
\end{aligned} \tag{2}$$

Another of Masuda's results (his Theorem 3) states that, under the assumption that $b_i > w_i$ for all i, every optimal solution of (1) has

$$\sum_{i=1}^{n} b_i y_i - \sum_{i=1}^{n} b_i z_i = d. \tag{3}$$

Under a relaxed assumption, we shall derive a conclusion which is strong enough for our purpose.

Lemma 1. *If $b_i > w_i$ for at least one i, then* (2) *has an optimal solution satisfying* (3).

Proof. Consider a counterexample and let x_i^*, y_i^*, z_i^*, t^* be an optimal solution of (2) which maximizes $\sum b_i y_i - \sum b_i z_i$ among all the optimal solutions of (2). We claim that

$$z_i^* = 0 \quad \text{for all } i. \tag{4}$$

Assume the contrary, so that $z_i^* > 0$ for some i. Now $y_i^* z_i^* = 0$ implies $y_i^* = 0$, whereupon $w_i x_i^* + b_i y_i^* - b_i z_i^* = d$ implies $x_i^* > 0$. But then the value of $\sum b_i y_i - \sum b_i z_i$ can be increased by the substitution $z_i^* \leftarrow z_i^* - \varepsilon$, $x_i^* \leftarrow x_i^* - b_i \varepsilon / w_i$ with a suitable positive ε, a contradiction. Thus (4) is verified.

It will be convenient to assume, without loss of generality, that

$$w_i < b_i \quad \text{whenever } 1 \le i \le k,$$
$$w_i \ge b_i \quad \text{whenever } k < i \le n,$$

and that w_1 is the smallest of the numbers $w_1, w_2, \ldots, w_k$. Now consider the linear programming problem

$$\begin{aligned} &\text{maximize} \quad \sum_{i=1}^{k} b_i y_i \\ &\text{subject to} \quad x_i + y_i \le t^* \qquad (i = 1, 2, \ldots, k), \\ &\qquad w_i x_i + b_i y_i = d \qquad (i = 1, 2, \ldots, k), \\ &\qquad \sum_{i=1}^{k} y_i \le t^*, \\ &\qquad \sum_{i=1}^{k} b_i y_i \le d, \\ &\qquad x_i,\ y_i \ge 0 \qquad (i = 1, 2, \ldots, k). \end{aligned} \tag{5}$$

By virtue of (4), this problem has at least one feasible solution; furthermore, every feasible solution of (5) can be extended into an optimal solution of (2) by setting $z_i = 0$ for all i and $x_i = d/w_i$, $y_i = 0$ whenever $k < i \le n$. Thus the optimal value of (5) is less than d.

Finally, if $\bar{x}_i$, $\bar{y}_i$ is an optimal solution of (5), then

$$d > \sum_{i=1}^{k} b_i \bar{y}_i \ge b_1 \bar{y}_1 + w_1 \sum_{i=2}^{k} \bar{y}_i = d - w_1 \bar{x}_1 + w_1 \sum_{i=2}^{k} \bar{y}_i,$$

and so

$$\bar{x}_1 > \sum_{i=2}^{k} \bar{y}_i.$$

Now it follows that

$$\bar{x}_1 > 0 \quad \text{and} \quad \sum_{i=1}^{k} \bar{y}_i < \bar{x}_1 + \bar{y}_1 \le t^*.$$

But then the value of the objective function in (5) can be increased by the substitution $\bar{x}_1 \leftarrow \bar{x}_1 - \varepsilon$, $\bar{y}_1 \leftarrow \bar{y}_1 + w_1\varepsilon/b_1$ with a suitable positive ε. This contradiction completes the proof.

Now we are ready for a crucial step in the proof of (i).

Lemma 2. *If $b_i > w_i$ for at least one i, then there are (possibly different) positions $p_1, p_2, \ldots, p_n$ and a schedule transporting the bicycle from p_1 to $p_1 + d$ and each person k from p_k to $p_k + d$ in time t^*.*

Proof. By Lemma 1, some optimal solution x_i, y_i, z_i, t^* of (2) satisfies (3). For all $k = 1, 2, \ldots, n+1$, write

$$t_k = \sum_{i=1}^{k-1} (y_i + z_i), \qquad q_k = \sum_{i=1}^{k-1} b_i(y_i - z_i)$$

and

$$p_k = \begin{cases} q_k - w_k x_k & \text{if } x_k \le t_k, \\ q_k - w_k t_k & \text{if } x_k > t_k. \end{cases}$$

In the desired schedule, person k

walks from p_k	to q_k	in the first	t_k	units of time,
bicycles from q_k	to q_{k+1}	in the next	$t_{k+1} - t_k$	units of time,
walks from q_{k+1}	to $p_k + d$	in the remaining	$t^* - t_{k+1}$	units of time.

Theorem 1. *For every positive ε there is a schedule in which everybody arrives at the destination before time $t^* + \varepsilon$.*

Proof. If $b_i \le w_i$ for all i, then the conclusion is trivial: everybody simply walks to the destination. If $b_i > w_i$ for at least one i, then we can rely on Lemma 2. Let p be the largest of the numbers $|p_1|, |p_2|, \ldots, |p_n|$, let s be the smallest of the numbers $b_1, w_2, w_3, \ldots, w_n$ and let m be an integer greater than $2p/\varepsilon s$. Changing the time and distance scales in Lemma 2, we obtain a schedule transporting the bicycle from p_1/m to $(p_1 + d)/m$ and each person k from p_k/m to $(p_k + d)/m$ in time t^*/m. Repetitions of this schedule enable us to transport the bicycle from p_1/m to $p_1/m + d$ and each person k from p_k/m to $p_k/m + d$ in time t^*. Now the desired schedule goes as follows. First, in less than $\frac{1}{2}\varepsilon$ units of time, person 1 bicycles from 0 to p_1/m and each remaining person k walks from 0 to p_k/m. Then, in precisely t^* units of time, this configuration is shifted forward by distance d. Finally, in less than $\frac{1}{2}\varepsilon$ units of time, person 1 bicycles from $p_1/m+d$ to d and each remaining person k walks from $p_k/m+d$ to d.

Theorem 2. *If* $n=4$, $d=90$ *and*

$$w_1=w_2=13, \quad b_1=b_2=27, \quad w_3=w_4=3, \quad b_3=b_4=18,$$

then $t^*=10$ *but there is no schedule with the last person arriving in time* 10.

Proof. To see that $t^*\le 10$, consider the feasible solution

$$x_1=x_2=9, \qquad z_1=z_2=1, \qquad x_3=x_4=6, \qquad y_3=y_4=4, \qquad t=10 \tag{6}$$

of (1). To see that $t^*\ge 10$, sum up

the time constraint of person 1 multiplied by 78,
the time constraint of person 2 multiplied by 78,
the time constraint of person 3 multiplied by 33,
the time constraint of person 4 multiplied by 33,
the position constraint of person 1 multiplied by −6,
the position constraint of person 2 multiplied by −6,
the position constraint of person 3 multiplied by −11,
the position constraint of person 4 multiplied by −11,
the time constraint of the bicycle multiplied by 3,
the position constraint of the bicycle multiplied by 9.

Since the resulting inequality reads

$$156(u_1+u_2)+66(u_3+u_4)+162(y_1+y_2)+72(z_3+z_4)\le 225t-2250,$$

we conclude that every feasible solution of (1) has $t\ge 10$. In fact, we conclude that every feasible solution with $t=10$ must have

$$u_1=u_2=u_3=u_4=y_1=y_2=z_3=z_4=0,$$

and satisfy all the time constraints as equalities. Now it follows that (6) is the unique optimal solution of (1).

Next, consider a hypothetical schedule with the last person arriving in time 10. Since (6) is the unique optimal solution of (1),

person 1 must spend 1 unit of time bicycling backwards,
person 2 must spend 1 unit of time bicycling backwards,
person 3 must spend 4 units of time bicycling forward,
person 4 must spend 4 units of time bicycling forward.

But then the bicycle cannot have a first user: since $b_i > w_j$ for all i and j, this user would remove it from the rest of the team, and subsequently abandon it without the next user ready to take over.

If we restricted ourselves to schedules in which the bicycle always has a first user, our proof could end here. However, in some of Masuda's schedules the bicycle changes users infinitely many times in such a way that each user is followed by another. Since schedules without a last user are admissible, one might argue that schedules without a first user ought to be admissible as well. Rather than questioning this point of view, we shall proceed to show that Theorem 2 holds even under a quite broad interpretation of the term 'schedule'. The proof, involving elementary notions of real analysis, takes up the rest of this paper.

In our definition of a schedule, the trajectory of the bicycle is represented by a function f_0, the trajectory of the ith person is represented by a function f_i and the set of moments at which the ith person is riding the bicycle is denoted by E_i. Thus

$$\begin{aligned} &f_i(0)=0 &&\text{for all } i=0,1,\dots,n,\\ &f_i(t)=d &&\text{for all } i=1,2,\dots,n,\\ &f_i(s)=f_0(s) &&\text{whenever } s\in E_i, \end{aligned}$$

and all the pairwise intersections of $E_1, E_2, \dots, E_n$ have measure zero. We assume that each E_i is measurable, let $m_i(s)$ stand for the measure of $E_i \cap [0,s]$ and set $n_i(s) = s - m_i(s)$ whenever $1 \le s \le t$. Now

$$|f_0(s)-f_0(r)| \le \sum_{i=1}^{n} b_i(m_i(s)-m_i(r))$$

and

$$|f_i(s)-f_i(r)| \le b_i(m_i(s)-m_i(r)) + w_i(n_i(s)-n_i(r))$$

whenever $1 \le r \le s \le t$ and $i = 1, 2, \dots, n$. No other properties are required of the functions $f_0, f_1, \dots, f_n$ and the sets $E_1, E_2, \dots, E_n$. (This definition encompasses a rather large class of 'schedules': in particular, the bicycle may switch users uncountably many times.)

Again, where minimization of t is concerned, no generality is lost by assuming that each of the functions $f_0, f_1, \dots, f_n$ maps $[0,t]$ into $[0,d]$. Similarly, the link with the linear programming problem (1) extends to the context of these general schedules: it is an easy exercise (whose details we omit) to express each f_i with $i = 1, 2, \dots, n$ as a sum $g_i + h_i$ such that

$$\begin{aligned} &g_i(0)=h_i(0)=0,\\ &|g_i(s)-g_i(r)| \le b_i(m_i(s)-m_i(r)),\\ &|h_i(s)-h_i(r)| \le w_i(n_i(s)-n_i(r)),\\ &\sum_{i=1}^{n} g_i(s)=f_0(s), \end{aligned}$$

whereupon it is a routine matter to verify that the numbers x_i, u_i, y_i, z_i defined by

$$x_i=\frac{1}{w_i}h_i^+(t),\qquad u_i=\frac{1}{w_i}h_i^-(t),\qquad y_i=\frac{1}{b_i}g_i^+(t),\qquad z_i=\frac{1}{b_i}g_i^-(t)$$

constitute a feasible solution of (1).

In particular, if Theorem 2 is false then, since (6) is the unique optimal solution of (1), there is a counterexample with

$$g_1(10) = g_2(10) = -27, \qquad g_3(10) = g_4(10) = 72$$

and

$$h_1(10) = h_2(10) = 117, \qquad h_3(10) = h_4(10) = 18.$$

But then the inequalities $|g_i(10)| \le b_i m_i(10)$ and $\sum m_i(10) \le 10$ imply $m_1(10) = m_2(10) = 1$ and $m_3(10) = m_4(10) = 4$. Hence

$$m_1(s) + m_2(s) + m_3(s) + m_4(s) = s \tag{7}$$

for all s. Furthermore, since

$$|g_i(10)| \le |g_i(s)| + |g_i(10) - g_i(s)|,$$
$$|g_i(s)| \le b_i m_i(s),$$
$$|g_i(10) - g_i(s)| \le b_i (m_i(10) - m_i(s))$$

and

$$|g_i(10)| = b_i m_i(10),$$

each $g_i(s)$ must have the sign of $g_i(10)$ and the magnitude of $b_i m_i(s)$. Explicitly,

$$g_1(s) = -27m_1(s), \qquad g_2(s) = -27m_2(s),$$
$$g_3(s) = 18m_3(s), \qquad g_4(s) = 18m_4(s).$$

An analogous argument shows that

$$h_1(s) = 13n_1(s), \qquad h_2(s) = 13n_2(s),$$
$$h_3(s) = 3n_3(s), \qquad h_4(s) = 3n_4(s).$$

Hence

$$\begin{aligned} f_0(s) &= -27m_1(s) - 27m_2(s) + 18m_3(s) + 18m_4(s), \\ f_1(s) &= 13s - 40m_1(s), \\ f_2(s) &= 13s - 40m_2(s), \\ f_3(s) &= 3s + 15m_3(s), \\ f_4(s) &= 3s + 15m_4(s). \end{aligned} \tag{8}$$

Our next two remarks apply generally to optimal schedules in which the bicycle is always in use (except for a set of measure zero) and moving faster than all the walking speeds w_i. First, let us call a point s in time *singular* if all the $n+1$ positions coincide. Note that these positions must coincide at sd/t, for otherwise the restriction of this schedule onto $[0,s]$ or $[s,t]$ could be transformed into a schedule requiring less than the optimal time t. Furthermore, note that the set of all singular points is closed and that its interior is empty. Hence there must be distinct singular points r and s such that no x with $r<x<s$ is singular. But then the restriction of our schedule onto $[r,s]$ can be transformed into an optimal schedule with no singular

points except 0 and t. Thus, where minimization of t is concerned, no generality is lost by assuming that 0 and t are the only singular points. Secondly, consider a point r in time such that precisely two positive subscripts k have $f_k(r)=f_0(r)$. We claim that there is a positive subscript p such that, for some positive δ and for every s such that $r<s<r+\delta$, we have $f_q(s)\neq f_0(s)$ whenever $q>0$ and $q\neq p$. In fact, with i and j standing for the two positive subscripts such that $f_i(r)=f_j(r)=f_0(r)$, it suffices to choose δ so that each $f_k(s)$ with $k\neq i, j, 0$ and $r<s\leq r+\delta$ is different from $f_i(s)$, $f_j(s)$ and $f_0(s)$. To verify this claim, note that the set of all the points s with $r\leq s\leq r+\delta$ and $f_i(s)=f_j(s)$ is closed and has an empty interior; it will suffice to show that this set contains no point other than r. Assuming the contrary, observe that there are points s_1, s_2 with

$$r\leq s_1<s_2\leq r+\delta, \qquad f_i(s_1)=f_j(s_1), \qquad f_i(s_2)=f_j(s_2)$$

and

$$f_i(s)\neq f_j(s) \quad \text{whenever } s_1<s<s_2.$$

Now the open sets A_i, A_j defined by

$$A_k=\{s: s_1<s<s_2, f_k(s)\neq f_0(s)\}$$

are disjoint and their union is the open interval (s_1, s_2). Hence one of these sets, say A_j, must be empty. But then

$$|f_j(s_2)-f_j(s_1)|>w_i(s_2-s_1) \quad \text{and} \quad |f_i(s_2)-f_i(s_1)|\leq w_i(s_2-s_1),$$

a contradiction.

The first of these remarks guarantees that no generality is lost by assuming that there are no singular points other than 0 and t. Note, however, that there must be at least one point r in time such that $0<r<t$ and $f_k(r)=f_0(r)$ for at least two positive subscripts k. Using (7) and (8), we can classify all such points r into the following four categories, with persons 1, 2 referred to as *fast* and persons 3, 4 referred to as *slow:*

(i) The two fast persons and the bicycle are in the same position, with one slow person ahead and the other slow person behind by the same distance $3x$.

(ii) One fast person, one slow person and the bicycle are in the same position, with the remaining two persons ahead, the fast person at a distance $4x$ and the slow person at a distance $6x$ from the bicycle.

(iii) The two slow persons and the bicycle are in the same position, with one fast person ahead and the other fast person behind by the same distance $2x$.

(iv) One fast person, one slow person and the bicycle are in the same position, with the remaining two persons behind, the fast person at a distance $4x$ and the slow person at a distance $6x$ from the bicycle.

The second of the two remarks made above guarantees that only one of the two potential users will actually use the bicycle in the next δ units at time. Note that the user must be slow in case (ii) and fast in case (iv): otherwise the bicycle would be getting farther and farther away from the rest of the team. Now it becomes easy to verify that

$$\begin{array}{llll}\text{if } r \text{ is of type (i),} & \text{then } r+x/10 \text{ is of type (ii)} & \text{with the same } x, \\ \text{if } r \text{ is of type (ii),} & \text{then } r+2x/5 \text{ is of type (iii)} & \text{with the same } x, \\ \text{if } r \text{ is of type (iii),} & \text{then } r+2x/5 \text{ is of type (iv)} & \text{with the same } x, \\ \text{if } r \text{ is of type (iv),} & \text{then } r+x/10 \text{ is of type (i)} & \text{with the same } x \end{array}$$

and that, in all four cases,

none of the points s with $r \le s \le r+x$ is singular.

But then no point s with $s \ge r$ is singular, which is the desired contradiction.

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Reference

[1] S. Masuda, The bicycle problem, Research Report ORC 70-35, University of California, Berkeley, October 1970.